Olympiad

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Inequalities

Duration : 3h 30 min

Problem 1: (propoesd by SABIR Ilyass)

suppose $a_1, a_2, ..., a_n$ are positive real numbers with sum n ,Prove that :

$$\sum_{i=1}^{n} \sqrt{a_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_j^2} \geq n\sqrt{1+\sqrt{n-1}}$$

Problem 2: (propoesd by SABIR Ilyass)

Let α, a, x_1, x_2, ..., x_n be non − negative real numbers such that $x_1 \times x_2 \times \ldots \times x_n = \alpha^n$ Prove that:

$$\sum_{i=1}^{n} \frac{x_i^n}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (a + x_j)} \geq \frac{n\alpha^n}{(a+\alpha)^{n-1}}$$

Problem 3: (propoesd by SABIR Ilyass)

let $a_1, a_2, ..., a_n$ are positive real numbers , Prove that :

$$\forall l \in [\![1,n]\!] , \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_l=1}^{n} \frac{1}{\sum\limits_{\substack{j=1 \\ j \neq i_1,\ldots,i_l}}^{n} a_j^n + \sum\limits_{k=1}^{l} a_{i_k}^l \prod\limits_{\substack{j=1 \\ j \neq i_1,\ldots,i_l}}^{n} a_j} \leq \frac{1}{\left(\prod\limits_{j=1}^{n} a_j\right)\left(\sum\limits_{i=1}^{n} a_i^l\right)} \left(\sum_{i=1}^{n} a_i\right)^l$$

SOLUTION

Problem 1:

Solution (proposed by SABIR Ilyass-SAFI,Morocco,14/08/2018)

According to Cauchy-schwarz inequality , we have :

$$\forall i \in [\![1,n]\!] \sqrt{\sum_{\substack{j=1\\j\neq i}}^{n} a_j^2} \geq \frac{\sum\limits_{\substack{j=1\\j\neq i}}^{n} a_j}{\sqrt{n-1}} = \frac{n-a_i}{\sqrt{n-1}}$$

we conclude that : $\forall i \in [\![2,n]\!], \forall k \in [\![1,i-1]\!]$

$$\sqrt{a_i+\sqrt{\sum_{\substack{j=1\\j\neq i}}^{n} a_j^2}}\sqrt{a_k+\sqrt{\sum_{\substack{l=1\\l\neq k}}^{n} a_l^2}} \geq \sqrt{a_i+\frac{n-a_i}{\sqrt{n-1}}}\sqrt{a_k+\frac{n-a_k}{\sqrt{n-1}}}$$

$$=\frac{1}{\sqrt{n-1}}\sqrt{(\sqrt{n-1}-1)a_i+n}\sqrt{(\sqrt{n-1}-1)a_k+n}$$

$$\geq \frac{1}{\sqrt{n-1}}((\sqrt{n-1}-1)\sqrt{a_ia_k}+n)$$

$$\Rightarrow \sum_{i=2}^{n}\sum_{k=1}^{i-1}\sqrt{a_i+\sum_{\substack{j=1\\j\neq i}}^{n} a_j^2}\sqrt{a_k+\sum_{\substack{l=1\\l\neq k}}^{n} a_l^2} \geq \frac{1}{\sqrt{n-1}}\sum_{i=2}^{n}\sum_{k=1}^{i-1}((\sqrt{n-1}-1)\sqrt{a_ia_k}+n)$$

$$=\left(1-\frac{1}{\sqrt{n-1}}\right)\sum_{i=2}^{n}\sum_{k=1}^{i-1}\sqrt{a_ia_k}+\frac{n^2\sqrt{n-1}}{2}$$

So, $$\left(\sum_{i=1}^{n}\sqrt{a_i+\sum_{\substack{j=1\\j\neq i}}^{n} a_j^2}\right)^2 = \sum_{i=1}^{n}\left(a_i+\sum_{\substack{j=1\\j\neq i}}^{n} a_j^2\right)+2\sum_{i=2}^{n}\sum_{k=1}^{i-1}\sqrt{a_i+\sum_{\substack{j=1\\j\neq i}}^{n} a_j^2}\sqrt{a_k+\sum_{\substack{l=1\\l\neq k}}^{n} a_l^2}$$

$$\geq \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} a_j^2+2\left(1-\frac{1}{\sqrt{n-1}}\right)\sum_{i=2}^{n}\sum_{k=1}^{i-1}\sqrt{a_ia_k}+n^2\sqrt{n-1}+n$$

$$=\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}a_j^2+\left(1-\frac{1}{\sqrt{n-1}}\right)\left(\left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2-\sum_{i=1}^{n}a_i\right)+n^2\sqrt{n-1}+n$$

$$=\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}a_j^2+\left(1-\frac{1}{\sqrt{n-1}}\right)\left(\left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2-\sum_{i=1}^{n}a_i\right)+n^2\sqrt{n-1}+n$$

$$=\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}a_j^2+\left(1-\frac{1}{\sqrt{n-1}}\right)\left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2+n^2\sqrt{n-1}+\frac{n}{\sqrt{n-1}}$$

After some computations , we find : $\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}a_j^2+\left(1-\frac{1}{\sqrt{n-1}}\right)\left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2\geq n^2-\frac{n}{\sqrt{n-1}}$

∇

Problem 2:

Solution (proposed by SABIR Ilyass-SAFI,Morocco,14/08/2018)

According to AM-GM inéquality ,we have : $\forall i\in[\![1,n]\!],\forall\boldsymbol{\beta}\succ 0\ \frac{x_i}{\prod_{\substack{j=1\\j\neq i}}^{n}(a+x_j)}+\sum_{\substack{i=1\\j\neq i}}^{n}\frac{a+x_j}{\boldsymbol{\beta}^n}\geq n\frac{x_i}{\boldsymbol{\beta}^{n-1}}$

$$\text{so ,}\quad \sum_{i=1}^{n}\frac{x_i}{\prod_{\substack{j=1\\j\neq i}}^{n}(a+x_j)}\geq\frac{1}{\boldsymbol{\beta}^{n-1}}\left(n-\frac{n-1}{\boldsymbol{\beta}}\right)\sum_{i=1}^{n}x_i-\frac{\text{an}(n-1)}{\boldsymbol{\beta}^n}\quad ,\forall\boldsymbol{\beta}\succ 0$$

$$\geq\frac{n\alpha}{\boldsymbol{\beta}^{n-1}}\left(n-\frac{n-1}{\boldsymbol{\beta}}\right)-\frac{\text{an}(n-1)}{\boldsymbol{\beta}^n}\quad ,\forall\boldsymbol{\beta}\succ 0$$

the function $\varphi\colon\boldsymbol{\beta}\succ 0\longmapsto\frac{n\alpha}{\boldsymbol{\beta}^{n-1}}\left(n-\frac{n-1}{\boldsymbol{\beta}}\right)-\frac{\text{an}(n-1)}{\boldsymbol{\beta}^n}$ than we have:

$$\forall\boldsymbol{\beta}\succ 0\ \varphi'(\boldsymbol{\beta})=\frac{n^2(n-1)}{\boldsymbol{\beta}^{n+1}}(\alpha(1-\beta)-a)$$

Clearly φ' is $\searrow$,and $\varphi'(\boldsymbol{\beta})=0\Leftrightarrow\boldsymbol{\beta}=\frac{a+\alpha}{\alpha}$ so :

$\max_{t\succ 0}\varphi(t)=\frac{n\alpha}{(\frac{a+\alpha}{\alpha})^{n-1}}\left(n-\frac{n-1}{\frac{a+\alpha}{\alpha}}\right)-\frac{\text{an}(n-1)}{(\frac{a+\alpha}{\alpha})^n}=\frac{n\alpha^n}{(a+\alpha)^{n-1}}$ the problem is completely solved

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Problem 3:

Solution(proposed by SABIR Ilyass SAfi,Morocco,17/08/2018):

According to AM-GM inequality,we have :

$$\forall i_1,....,i_l\in[\![1,n]\!]\ \mathrm{l}a_i^n+\sum_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j^n \geq \mathrm{n}a_i^l \prod_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j$$

$$\Rightarrow\forall i_1,....,i_l\in[\![1,n]\!]\ \sum_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n}\left(a_i^n+\sum_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j^n\right)\geq n\sum_{\substack{i=1\\ i=i_1,....,i_l}}^{n} a_i^l \prod_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j$$

$$\Leftrightarrow\forall i_1,....,i_l\in[\![1,n]\!]\ \sum_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j^n\geq \sum_{\substack{i=1\\ i=i_1,....,i_l}}^{n} a_i^l \prod_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j$$

$$\Leftrightarrow\forall i_1,....,i_l\in[\![1,n]\!]\ \sum_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j^n+\sum_{k=1}^{l} a_{i_k}^l \prod_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j\geq \sum_{\substack{i=1\\ i=i_1,....,i_l}}^{n} a_i^l \prod_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j+\sum_{k=1}^{l} a_{i_k}^l \prod_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j$$

$$=\sum_{i=1}^{n} a_i^l \prod_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j$$

$$\text{so : }\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_l=1}^{n}\frac{\prod_{j=1}^{n} a_j}{\sum_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j^n+\sum_{k=1}^{l} a_{i_k}^l \prod_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j}\leq \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_l=1}^{n}\frac{\prod_{j=1}^{n} a_j}{\sum_{i=1}^{n} a_i^l \prod_{\substack{j=1\\ j\neq i_1,....,i_l}}^{n} a_j}$$

$$=\frac{1}{\sum_{i=1}^{n} a_i^l}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_l=1}^{n}\prod_{j=i_1,....,i_l}^{n} a_j$$

$$=\frac{1}{\sum_{i=1}^{n} a_i^l}\left(\sum_{i=1}^{n} a_i\right)^l$$

This ends the proof

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